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the lectures pdfs are available at:

https://www.physics.umd.edu/rgroups/amo/orozco/results/2022/Results22.htm

Correlations in Optics and Quantum Optics; A series of lectures about correlations and coherence 1. November 2022

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Tentative list of topics to cover:

- From statistics and linear algebra to power spectral densities
- Historical perspectives and examples in many areas of physics
- Correlation functions in classical optics (field-field; intensity-intensity; field-intensity)
- Correlation functions in quantum examples
- Correlations and conditional dynamics for control
- Correlations in quantum optics of the field and intensity
- Optical Cavity QED
- From Cavity QED to waveguide QED.

1. Some statistics

- Think of data as column vector or vectors, such that their mean is zero.
- Each point is (x_i) (x_i,y_i)
- We can use linear algebra.

Correlation of a set of data:

- Think of data as a column vector and assume that each point is $x_i = \bar{x} + \delta x_i$

- Calculate the variance of the data

- $Var(x) = \frac{\sum(x_i - \bar{x})^2}{N} = \frac{\sum(\bar{x} + \delta x_i - \bar{x})^2}{N} = \frac{\sum(\delta x_i)^2}{N}$

- The average of the square of deviations from the mean

- The internal product of the vector with itself, is proportional to the variance.

- The correlation is the normalized internal product (IP).

- $\mathrm{IP}(x) = \sum(\bar{x} + \delta x_i)(\bar{x} + \delta x_i) = N\bar{x}^2 + 2\,\bar{x}\sum \delta x_i + \sum(\delta x_i)^2$

Correlation of a set of data:

- Think of data as two column vectors, such that their mean is zero.
- Each point is (x_i, y_i)
- The correlation function *C* is the internal product of the two vectors normalized by the norm of the vectors.

$$C = \frac{\sum_i x_i y_i}{\left(\sum_i x_i^2 \sum_i y_i^2\right)^{1/2}} = \frac{\vec{x} \bullet \vec{y}}{|\vec{x}||\vec{y}|} = \vec{X} \bullet \vec{Y}$$

If $y_i=mx_i$ (assuming the mean of x is zero and the mean of y is zero)

$$C=\frac{\sum_i x_i y_i}{\left(\sum_i x_i^2 \sum_i y_i^2\right)^{1/2}}=\frac{\sum_i x_i m x_i}{\left(\sum_i x_i^2 \sum_i m^2 x_i^2\right)^{1/2}}$$

$$C=\frac{m\sum_i x_i^2}{\left(m^2\sum_i x_i^2 \sum_i x_i^2\right)^{1/2}}=\frac{m\vec{x}\bullet\vec{x}}{|m||\vec{x}||\vec{x}|}=\pm 1$$

C acquires the extreme values

C is the internal product between the output data vector (y) and the value of the expected (fit) function (f(x)) with the input data (x)

$$C = \frac{\sum_i f(x_i) y_i}{\left(\sum_i f(x_i)^2 \sum_i y_i^2 \right)^{1/2}} = \frac{\vec{f} \bullet \vec{y}}{|\vec{f}||\vec{y}|} = \vec{F} \bullet \vec{Y}$$

Note that there is no reference to error bars or uncertainties in the data points

This correlation coefficient can be between two measurements or a measurement and a prediction…

- C is bounded : $-1<C+1$
- C it is $cos(\phi)$ where ϕ is in some abstract space.
- Correlation does not imply causality!

Think of your data as vectors, it can be very useful.

Correlations are not limited to a single spatial or temporal point.

In continuous functions, such as a time series, the correlation depends on the difference between the two comparing times.

The correlation can depend on real distance, angular distance or on any other parameter that characterizes a function or series.

Least squares fit

Minimize the autocorrelation of the difference between the observed value x_i and the expected value $f(x_i)$ divided by the statistical error in the observed value e_i. The internal product is

$$\chi^2 = \sum \frac{(x_i - f(x_i))^2}{e_i^2} = DOF$$

If the errors are random then you expect them to be Gaussian distributed, and the sum should be DOF

$$\chi^2_{red} = \frac{\chi^2}{DOF}$$

with variance

$$Var(\chi^2_{red}) = \frac{2}{DOF}$$

$$\chi^2_{red} = \frac{\chi^2}{DOF} \pm \sqrt{\frac{2}{DOF}}$$

The least squares problem can be formulated as finding the best solution to a set of m equations that has only n variables ($m>n$), more rows than columns.

see e. g.: Gilbert Strang, Introduction to Linear Algebra Ch. 4

1 Solving $\boxed{A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}}$ gives the projection $\boldsymbol{p} = A\widehat{\boldsymbol{x}}$ of $\boldsymbol{b}$ onto the column space of A.

2 When $A\boldsymbol{x} = \boldsymbol{b}$ has no solution, $\widehat{\boldsymbol{x}}$ is the "least-squares solution": $||\boldsymbol{b} - A\widehat{\boldsymbol{x}}||^2 =$ minimum.

3 Setting partial derivatives of $E = ||A\boldsymbol{x} - \boldsymbol{b}||^2$ to zero $\left(\frac{\partial E}{\partial x_i} = 0\right)$ also produces $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$.

4 To fit points $(t_1, b_1), \ldots, (t_m, b_m)$ by a straight line, A has columns $(1, \ldots, 1)$ and $(t_1, \ldots, t_m)$.

5 In that case $A^{\mathrm{T}}A$ is the 2 by 2 matrix $\begin{bmatrix} m & \Sigma\, t_i \\ \Sigma\, t_i & \Sigma\, t_i^2 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b}$ is the vector $\begin{bmatrix} \Sigma\, b_i \\ \Sigma\, t_i b_i \end{bmatrix}$.

Beyond equal indices (time, position, …)

Cross correlation (two functions)
Autocorrelation (same function)

C(n)= $$(f \star g)[n] \overset{\mathrm{def}}{=} \sum_{m=-\infty}^{\infty} f^*[m]\, g[m+n].$$

C(τ)= $$(f \star g)(\tau) \overset{\mathrm{def}}{=} \int_{-\infty}^{\infty} f^*(t)\, g(t+\tau)\, dt,$$

Resembles the convolution between two functions

Some mathematical properties (there are generalizations to matrices)

The correlation of f(t) and g(t) is equivalente to the convolution of f*(-t) and g(t)

$$f \star g = f^*(-t) * g = f^* * g(-t).$$

If f is Hermitic, then the correlation and the convolution are the same.

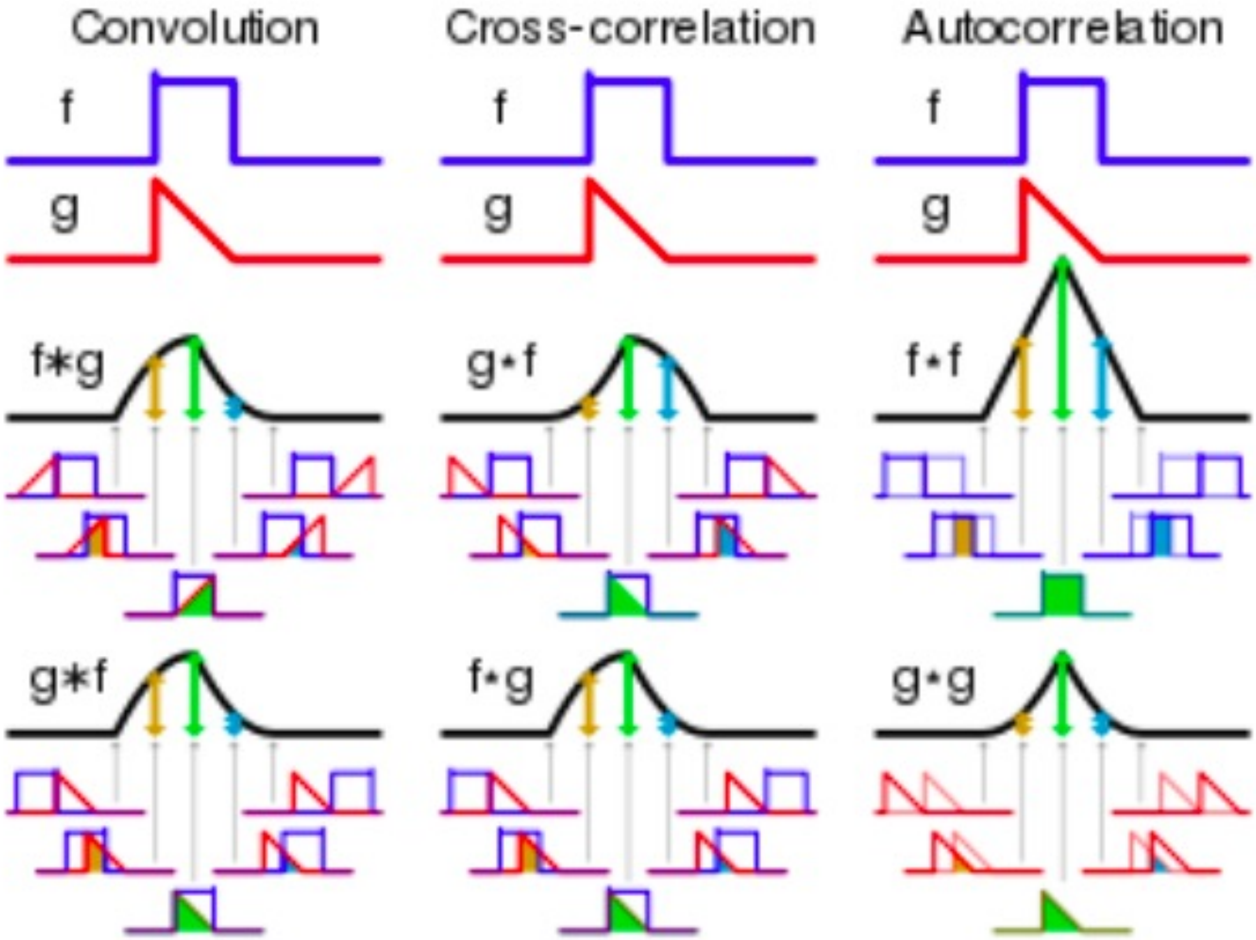
$$f \star g = f * g.$$

If both are Hermitic then: so

$$f \star g = g \star f.$$

Just as the convolution, the Fourier Transform:

$$\mathcal{F}\{f \star g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}^*$$


Convolution
Cross-correlation
Autocorrelation
f
g
f
g
f
g
f∗g
g⋆f
f⋆f
g∗f
f⋆g
g⋆g

The correlation function contains averaging, and you could think of it as some monent over a distribution:

$$C(\tau)=<x(t)x(t+\tau)>$$

Where the probability density has to satisfy the properties of a positivity, integral equal to one…

Noise

A note by Einstein:
Method for the determination of the statistical values of observations concerning quantities subject to irregular fluctuations.

Printed version of a lecture delivered on 28 February 1914 to a meeting of the Société suisse de physique in Basel.

PUBLISHED 15 March 1914

IN: *Archives des sciences physiques et naturelles* 37 (1914): 254–256.

A. Einstein (Zurich). — *Méthode pour la détermination de valeurs statistiques d'observations concernant des grandeurs soumises à des fluctuations irrégulières.* [1]

Supposons que la quantité $y = F(t)$ (par exemple le nombre des taches solaires), soit déterminée empiriquement en fonction du temps, pour un intervalle très grand, T. Comment peut-on représenter l'allure statistique de y?

Une réponse à cette question, suggérée par la théorie du rayonnement, est la suivante.

On suppose y développé en série de Fourier : [2]

$$y = F(t) = \sum A_n \cos \pi n \frac{t}{T}$$

Les cœfficients successifs A_n du développement seront, en grandeur et en signe, très différent les uns des autres et se succèderont de façon irrégulière. Mais si l'on forme la valeur moyenne $\overline{A^2_n}$ de A^2_n pour un intervalle Δn de n très grand, mais cependant suffisamment petit pour que $\frac{\pi \Delta n}{T}$ soit très petit, cette valeur moyenne sera une fonction continue de n.

Nous l'appellerons l'*intensité* I de y correspondant à n. L'intensité ainsi définie aura une période $\theta = \frac{T}{n}$; nous la désignerons par I (θ); le problème consiste à la déterminer.

[3] Un calcul simple donne :

$$I(\theta) = \overline{A^2_n} = \frac{2}{T^2}\int_0^T\int_0^T F(t)F(n) \,.\, \cos \pi n \,\frac{t - n}{T}\, dndt$$

Il suit de là que la fonction I cherchée peut être déterminée, à un facteur numérique près, par la règle suivante :

On choisit un intervalle de temps Δ et on forme la valeur moyenne :

$$\mathfrak{M}(\Delta) = \overline{F(t)F(t + \Delta)} \tag{1}$$

qui, pour la courbe donnée y, est une fonction caractéristique de Δ. Cette courbe tendra, pour de grandes valeurs de Δ, vers une limite, que l'on pourra rendre nulle par une translation convenable de l'axe des abcisses (axe des t et axe des Δ). Alors on a :

[4] (2)

$$I(\theta) = \int_0^\infty \mathfrak{M}(\Delta) \cos \pi \frac{\Delta}{\theta} \, d\Delta$$

Pour effectuer l'intégration indiquée en (2), on connaît déjà des dispositifs mécaniques. Mon ami, M. P. Habicht, m'a montré en outre que la détermination des moyennes de (1) peuvent se faire aisément à l'aide d'un intégrateur mécanique de maniement facile. L'exécution pratique de la méthode semble donc n'offrir aucune difficulté particulière.

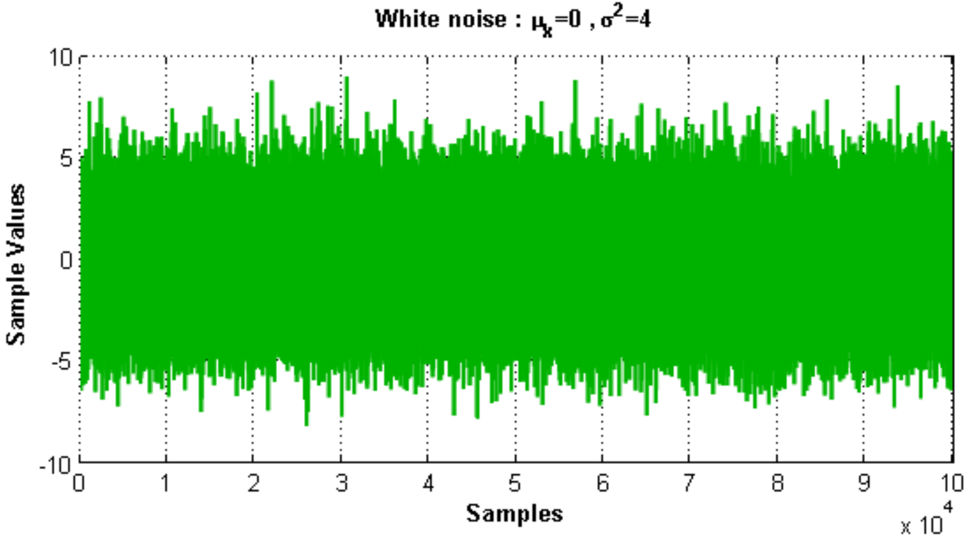
Nous ferons encore remarquer qu'un intégrateur permettant de former des moyennes du type (1), peut aussi être employé pour répondre à la question suivante : Y a-t-il ou non entre deux grandeurs F_1 et F_2 qui toutes deux sont déterminées empiriquement en fonction du temps, une relation de cause à effet ? Si l'on forme en effet,

$$\mathfrak{M}(\Delta) = \overline{F_1(t) F_2(t + \Delta)}$$

M. Weiss fait remarquer combien la méthode de M. Einstein rendra de services à la météorologie très riche en matériaux qui, jusqu'à présent, étaient à peu près inutilisables.

Told us to take the FT of the autocorrelation.

If you only have noise, there are formal problems to find the power spectral density, it is not a simple Fourier transform.



The Wiener–Khinchin-Kolmogorov theorem says that the power spectral density of noise is the Fourier transform of its autocorrelation.

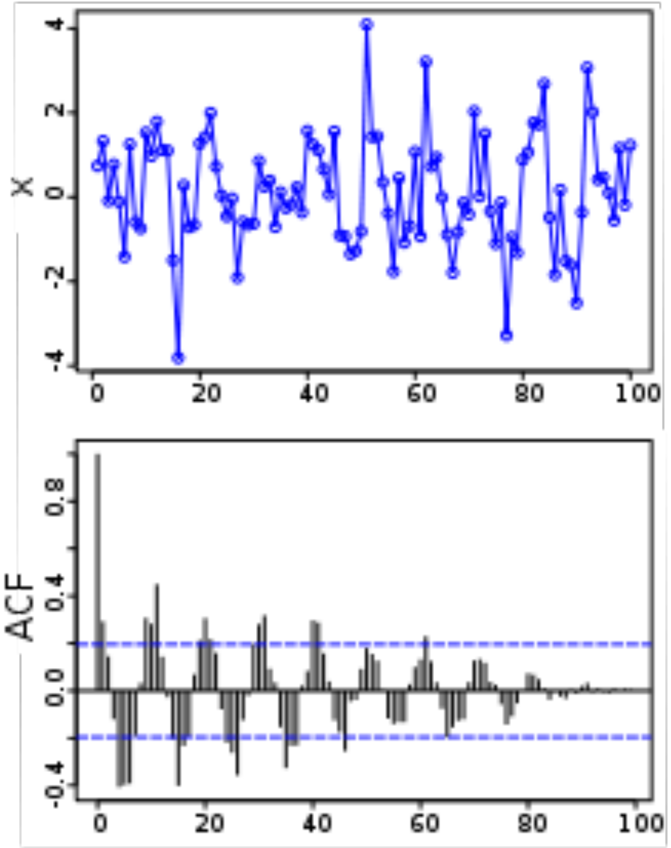
The Fourier transform of $f(t)$

$$g(\omega) = \int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t}$$

The power at a certain frequency: The Fourier transform of $|g(\omega)|^2$

$$\begin{aligned}
\int \frac{d\omega}{2\pi} |g(\omega)|^2 e^{-i\omega t} &= \int \frac{d\omega}{2\pi} g^*(\omega) e^{-i\omega t} \int dt'\, f(t') e^{i\omega t'} \\
&= \int dt'\, f(t') \int \frac{d\omega}{2\pi} g^*(\omega) e^{i\omega t'} e^{-i\omega t} \\
&= \int dt'\, f(t') \left[\int \frac{d\omega}{2\pi} g(\omega) e^{-i\omega(t'-t)} \right]^* \\
&= \int dt'\, f(t') f(t'-t)^*
\end{aligned}$$

The blue signal has a sinusoidal only visible in the autocrrelation (black)



Correlation as filter

A filter for data (1).

If you want to smooth a signal: take the data vector x_j with length m and calculate the sliding inner product with the filter vector f_i of length n, with $n<m$

A filter for data (2).

The simplest filter is all 0 and only k elements 1.
Aligning the two vectors at one end i=j=0 to i=n and j=n obtaining C_0
Move the filter vector over the data by one i=0 j=1 to i=n and j=n+1 assigning that to C_1, continue the process until you reach C_m
C(m) will be the signal smoothed by an average of k events.
It is widely used in image processing.

Consider a 1D image as a vector of numbers

5	4	2	3	7	4	6	5	3	6

Make an average with the neighbors, beware of the extremes. For example, for the fourth pixel with value 3 we replace it with the value (2 + 3 + 7) / 3 = 4; For the third pixel we replace the value 2 with (4+2+3)/3=3

For the extremes repeat the last value

.	.	.	*5*	*5*	5	4	2	3	7	4	6	5	3	6	*6*	*6*	.	.	.

The correlation is then:

.	.	*5*	$\boxed{5}$	4	2	3	7	4	6	5	3	6	*6*	*6*	.	.	.

* * *

1/3	$\boxed{1/3}$	1/3

=

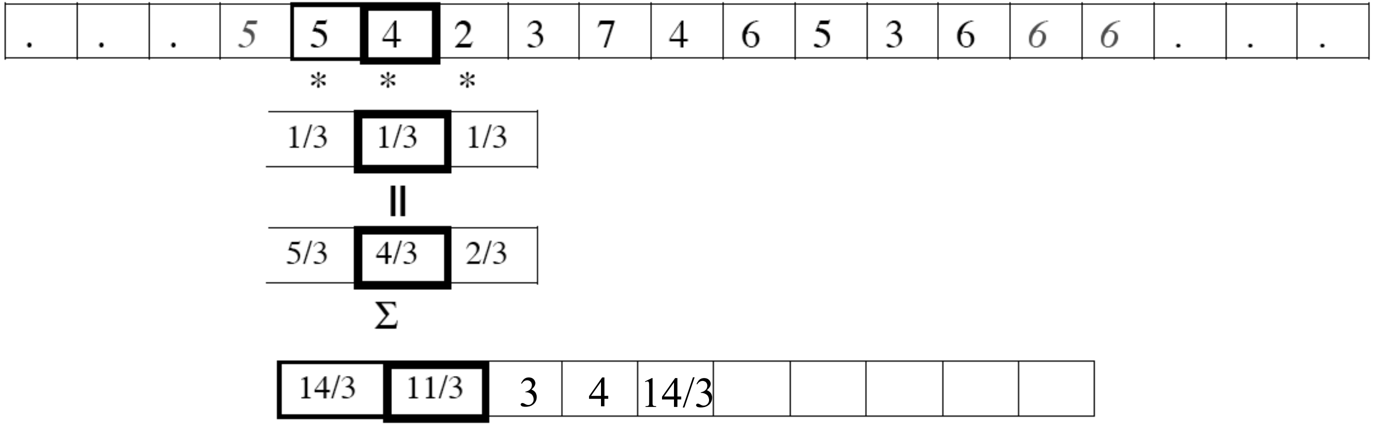
5/3	$\boxed{5/3}$	4/3

Σ

J

$\boxed{14/3}$									

The next value is:



We have a very simple filter, but you can define other more sophisticated even derived and integral.

Pattern to find (ignoring extremes)

3	7	5

Series

3	2	4	1	3	8	4	0	3	8	0	7	7	7	1	2

Normalized correlation

.946	.877	.934	.73	.81	.989	.64	.59	.78	.835	.61	.931	.95	.83	.57	.988

There are much more sophisticated forms but in principle they are similar

A filter for data (3).

If you known the form of the signal: take the data vector x_j with length m and calculate the inner product with the vector of the signal if of length n, with $n<m$

Start by aligning the two vectors at one end $i=j=0$ to $i=n$ and $j=n$ obtaining C_0

Then move the signal vector over the data one $i=0$ $j=1$ to $i=n$ and $j=n+1$ assigning that to C_1, continue the process until you reach C_m
$C(m)$ will have a maximum value when the signal pattern and data match, the noise will be averaged to zero.

- Correlations can be in time, in space, in angle, in pixels, ...
- The correlation can be auto (the function with itself) or cross (two functions).
- It can be between more than two vectors.
- It is a unique tool for studying and characterizing noise.

Thanks

given $\vec{y}, \vec{x}$

there may be a function (fit) such that $y_i = f(x_i)$

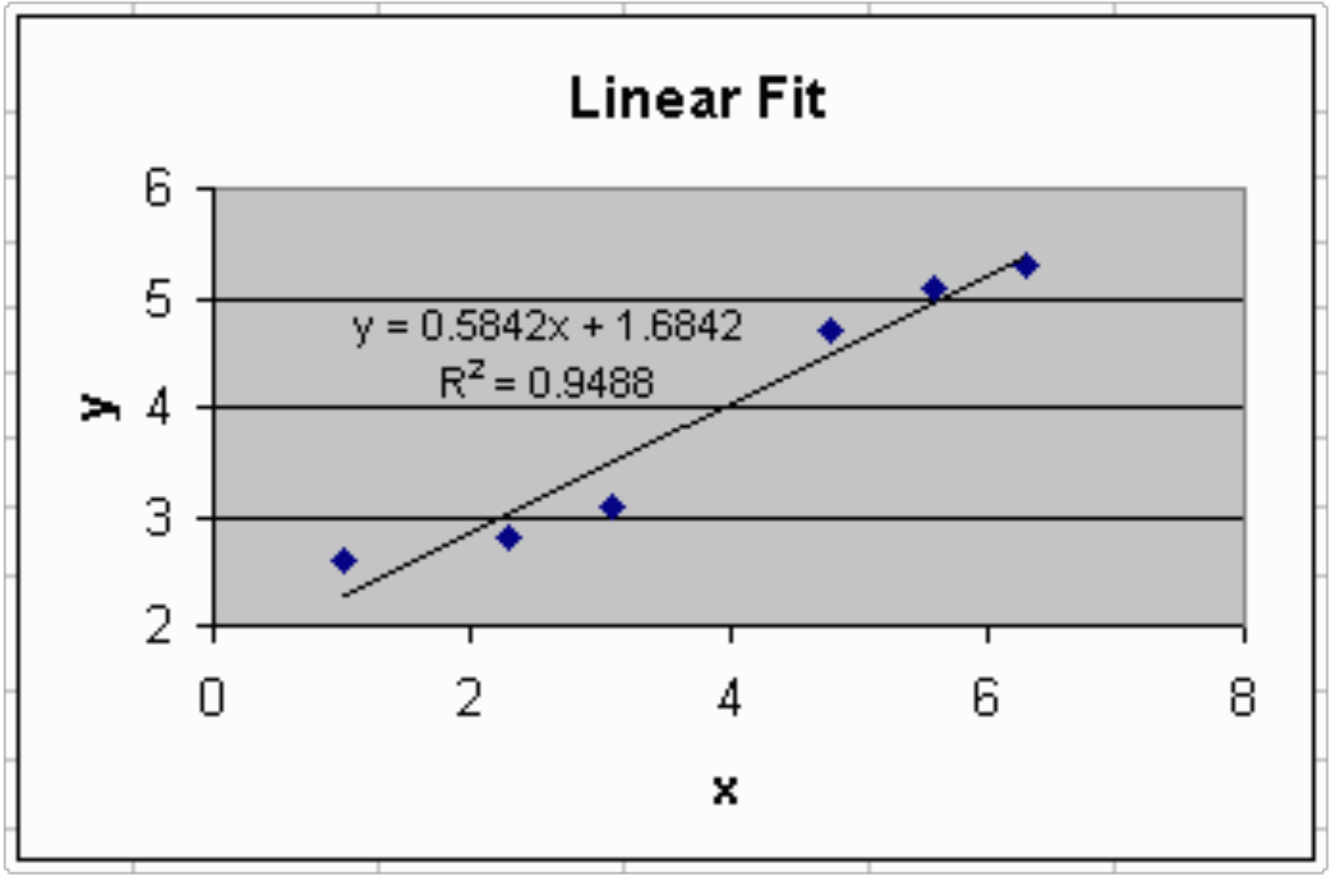
$$\bar{y} = \frac{1}{n}\sum_i y_i, \bar{x} = \frac{1}{n}\sum_i x_i$$

$$Var(y) = \frac{1}{(n-1)}\sum_i (y_i - \bar{y})^2, Var(x) = \frac{1}{(n-1)}\sum_i (x_i - \bar{x})^2$$

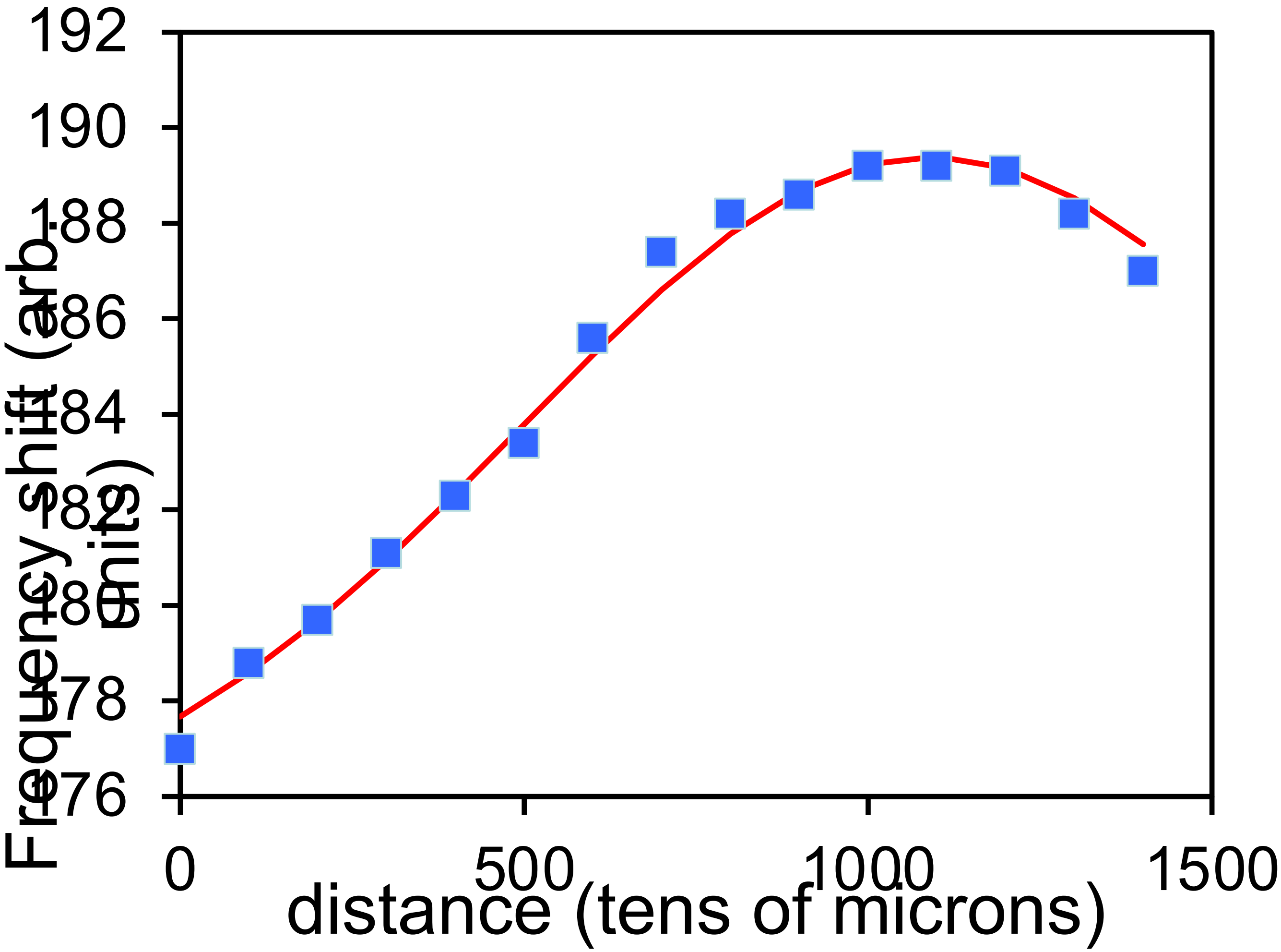
$$Var\ Exp(y) = \frac{1}{(n-1)}\sum_i (f(x_i) - \bar{y})^2$$

$$Var\ no\ Exp(y) = \frac{1}{(n-1)}\sum_i (f(x_i) - y_i)^2$$

$$C = \left(1 - \frac{Var\ no\ Exp}{Var}\right)^{1/2}$$



Correlation coefficient in this case C=R


192
190
188
186
184
182
180
178
176
Frequency shift (arb. units)
0
500
1000
1500
distance (tens of microns)

Data Vectors

X	Y	Y calc	(Y- Y calc)		XY	
0	177	177.67	-0.6736		31448.2	
100	178.8	178.59	0.20984		31931.9	
200	179.7	179.69	0.01373		32289.6	
300	181.1	180.95	0.15392		32769.3	
400	182.3	182.33	-0.0337		33239.4	
500	183.4	183.79	-0.3924		33707.5	
600	185.6	185.25	0.35335		34381.8	
700	187.4	186.61	0.79232		34970.3	
800	188.2	187.78	0.41799		35340.6	
900	188.6	188.68	-0.0818		35585.4	
1000	189.2	189.24	-0.0356		35803.4	
1100	189.2	189.4	-0.198		35834.1	
1200	189.1	189.16	-0.0552		35769.2	
1300	188.2	188.53	-0.3275		35480.9	
1400	187	187.57	-0.5665		35074.9	
			Least Squares			Correlation
∑Y^2	513552	513703	2.08244	∑YYc	513627	0.99999798